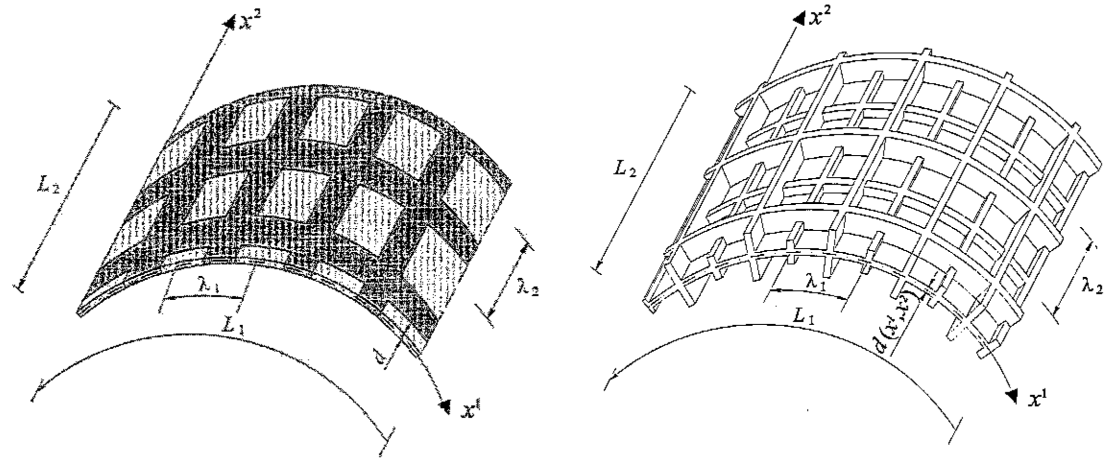
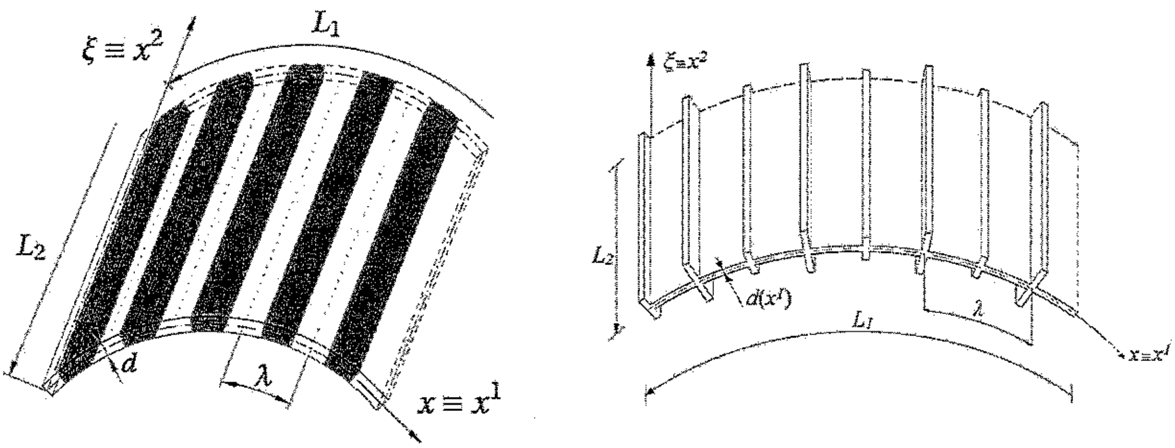
## SUMMARY

# Dynamics of Thin Micro-Periodic Cylindrical Shells: an Extended Version of the Tolerance Modelling Technique

The objects of considerations are thin linear-elastic Kirchhoff-Love-type circular cylindrical shells endowed with a material inhomogeneity and/or variable thickness and having a micro-periodic structure either in the circumferential and axial directions or only in the axial direction. Thus, we shall deal with either ***biperiodically or uniperiodically heterogeneous shells***. As examples we can mention cylindrical shells made of two kinds of periodically distributed materials or shells with periodically spaced families of thin stiffeners, cf. figures shown below. The period of heterogeneity is assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface. It means that the shells under consideration are composed of a large number of identical elements and every such element, called *a periodicity cell*, can be treated as a thin shell.



Examples of biperiodic shells



Examples of uniperiodic shells

The subject-matter of this doctoral dissertation is the analytical modelling of dynamic problems for the periodic shells under consideration and investigations of the effect of a cell size on the macroscopic and microscopic shell behaviour (*the length-scale effect*).

Dynamic behaviour of such shells are described by the known governing equations of Kirchhoff-Love theory for thin linearly elastic shells. For periodic shells, coefficients of these equations are highly oscillating, non-continuous and periodic functions. That is why the direct application of these equations to investigations of special dynamic problems is non-effective even using computational methods.

To obtain averaged equations with constant coefficients, a lot of different approximate modelling methods have been proposed. Periodic structures are usually described using *homogenized models* derived by means of *asymptotic methods.* These models represent certain equivalent structures with constant material properties. Unfortunately, *in models of this kind the effect of the microstructure size on the overall shell behaviour is neglected in the first approximation which is usually employed.*

An alternative (i.e. non-asymptotic) approach to the modelling of microheterogeneous media was proposed by Woźniak, cf. e.g. [**Woźniak Cz.** *et al.* (eds.): *Mathematical modelling and analysis in continuum mechanics of microstructured media.* Silesian University of Technology Press, Gliwice 2010]. This technique, called ***the tolerance modelling method***, is based on the concept of ***tolerance relations*** between points and real numbers related to the accuracy of the performed measurements and calculations. The tolerance relations are determined by ***the tolerance parameters.*** The other basic concepts of this procedure are those of ***slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation.*** The tolerance modelling is based on two assumptions. The first of them is called ***the tolerance averaging approximation*** and makes it possible to neglect terms of an order of tolerance parameters. The second one is termed ***the micro-macro decomposition.*** It states that the displacement fields occurring in the starting equations can be decomposed into ***unknown averaged displacements (macrodisplacements) being slowly-varying functions in periodicity directions and fluctuations represented by finite series of products of the known highly oscillating, continuous, periodic fluctuation shape functions and unknown slowly-varying fluctuation amplitudes.***

***A certain extended version of the known tolerance modelling technique has been proposed in*** [**Tomczyk B., Woźniak Cz.:** *Tolerance models in elastodynamics of certain reinforced thin-walled structures.* In: Kołakowski Z., Kowal-Michalska K. (eds.) *Statics, Dynamics and Stability of Structural Elements and Systems*, vol.2, Lodz University of Technology Press, Lodz, 123-153, **2012**]. ***This version is based on the new notion of weakly slowly-varying function, which is a certain extension of the well-known concept of slowly-varying function occurring in the classical tolerance modelling procedure. Both the weakly slowly-varying and the slowly-varying functions can be treated as constant on a periodicity***

*cell. The main difference between the weakly slowly-varying and the classical slowly-varying functions is that the products of derivatives of weakly slowly-varying functions in periodicity directions and microstructure length parameter (i.e. characteristic length dimension of the cell) are not assumed to be negligibly small. It means that the concept of a weakly slowly-varying function is less restrictive than the concept of a slowly-varying function. It also means that the averaged general model equations obtained by using the extended (general) tolerance modelling procedure contain a bigger number of terms dependent on the microstructure size than the averaged standard model equations derived by applying the classical tolerance modelling technique based on the notion of a slowly-varying function. The other basic concepts of the extended tolerance modelling technique as tolerance parameters, averaging operation, tolerance-periodic functions, fluctuation shape functions are the same as in the classical tolerance modelling procedure.* As the standard tolerance modelling, ***the general tolerance modelling*** is based on ***the tolerance averaging approximation*** and on ***the micro-macro decomposition,*** in which the unknown slowly-varying functions are replaced by the weakly slowly-varying functions.

***In the doctoral thesis, six mathematical averaged models with constant coefficients, constituting a proper tool for the analysis of selected dynamic problems in the thin linearly elastic biperiodic or uniperiodic cylindrical shells under consideration, were formulated and discussed. Moreover, four from these models make it possible to analyse the influence of a cell size on the dynamic shell behaviour (the length-scale effect). In order to formulate the models, the general (extended) tolerance modelling procedure, the consistent asymptotic modelling technique and the general (extended) combined asymptotic-tolerance modelling procedure have been applied to the starting Euler-Lagrange equations, which explicit form coincides with the governing equations of Kirchhoff-Love theory for thin linearly elastic cylindrical shells.***

***The two new general tolerance models*** for the analysis of selected dynamic problems in the biperiodic or uniperiodic shells under consideration have been formulated by applying ***the general tolerance modelling technique.*** This technique used to the starting Euler-Lagrange equations for biperiodic or uniperiodic shells has been realized in two steps. The first step has been based on ***the tolerance averaging of the starting lagrangian*** by applying ***the micro-macro decomposition, the averaging operation*** as well as ***the tolerance averaging approximation***. In the second step, using *the principle of stationary action* to ***the averaged action functional*** defined by means of *the* ***tolerantly averaged lagrangian***, we have arrived at the averaged Euler-Lagrange equations and then at their explicit form given by the constitutive relations and the dynamic balance equations. Summarizing, ***the two new general tolerance models for the analysis of dynamic problems for thin linearly elastic Kirchhoff-Love-type biperiodic or uniperiodic cylindrical are represented by the constitutive relations and the dynamic balance equations together with the micro-macro decomposition and the physical reliability***

***conditions***. The physical reliability conditions state that the basic unknowns called ***macrodisplacements*** and ***fluctuation amplitudes*** must be ***weakly slowly-varying*** either in the circumferential and axial directions (biperiodic shells) or only in the axial direction (uniperiodic shells). ***The resulting extended tolerance model equations have constant coefficients. Moreover, some of these coefficients depend on microstructure size. The length-scale effect can be analysed not only in dynamic, but also in stationary problems***.

From the comparison of the general tolerance models for biperiodic or uniperiodic shells, formulated here, with the corresponding known standard ones derived using the concept of slowly-varying function, cf. [**Tomczyk B.:** *Length-scale effect in dynamics and stability of thin periodic cylindrical shells*. Scientific Bulletin of the Lodz University of Technology, No. 1166, series: Scientific Dissertations, Lodz University of Technology Press, Lodz **2013**], it follows that the general and standard tolerance models have ***constant coefficients***. Moreover, ***some of them depend on a cell size***. However, ***governing equations of the general models contain a bigger number of terms dependent on the microstructure size than the standard model equations***. Thus, from the analytical results it follows that ***the general models make it possible to investigate the length-scale effect in more detail***. It has to be emphasized, that in the framework of ***the general tolerance model for <u>biperiodic shells</u>***, unknown fluctuation amplitudes are governed by a system of *partial differential equations* containing space and time derivatives of fluctuation amplitudes, whereas within *the standard tolerance model* these unknowns are governed by a system of *differential equations* involving only time derivatives of fluctuation amplitudes. It means that ***contrary* to *the general tolerance model equations*, *the standard tolerance model equations for the biperiodic shells under consideration don't make it possible to describe certain phenomena related to the micro-periodic structure of the shells, e.g. space-boundary layer phenomena strictly related to the specific form of boundary conditions imposed on unknown fluctuation amplitudes. These standard tolerance equations also don't describe the problems of propagation of displacement cell-dependent waves***. The governing equations of both the general and standard models for the considered biperiodically heterogeneous shells include terms with time derivatives of the fluctuation amplitudes. Hence, ***the equations of both models describe certain time-boundary layer phenomena strictly related to the specific form of initial conditions imposed on unknown fluctuation amplitudes***. On the other hand, the governing equations of both the general and standard models for the considered ***uniperiodically heterogeneous shells*** include terms with space and time derivatives of the fluctuation amplitudes. Hence, ***the equations of both uniperiodic shell models describe certain time-boundary and space-boundary layer phenomena***.

***The two mathematical consistent asymptotic models*** for the analysis of selected dynamic problems in the biperiodic or uniperiodic shells under consideration have been formulated by applying ***the new consistent asymptotic modelling procedure*** given in [**Woźniak Cz.** *et al.* (eds.): *Mathematical modelling and analysis in continuum mechanics of microstructured media*.

Silesian University of Technology Press, Gliwice 2010]. On passing from tolerance averaging to the consistent asymptotic averaging, the concepts of highly oscillating fluctuation shape functions and averaging operation are retained only. The notions of tolerance-periodic functions and slowly-varying or weakly slowly-varying functions are not introduced. The fundamental assumption imposed on the starting lagrangian in the framework of this approach is called ***the consistent asymptotic decomposition***. It states that the displacement fields occurring in the lagrangian have to be replaced by families of fields depending on parameter $\varepsilon \in (0, 1]$ and defined in an arbitrary cell. These families of displacements are decomposed into averaged part described by unknown functions (macrodisplacements) being continuously bounded in the periodicity directions/direction and highly oscillating part depending on ε. This highly oscillating part is represented by the known highly oscillating fluctuation shape functions multiplied by unknown functions (fluctuation amplitudes) being continuously bounded in the periodicity directions/direction.

Asymptotic modelling procedure applied to the starting Euler-Lagrange equations has been realized in two steps. The first step has been ***the consistent asymptotic averaging of the starting lagrangian under the consistent asymptotic decomposition***. In the second step, applying *the principle of stationary action* to *the consistent asymptotic action functional* defined by means of the asymptotically averaged lagrangian, we have arrived at the asymptotically averaged Euler-Lagrange equations and then at their explicit form. Finally, after eliminating unknown *fluctuation amplitudes*, we have obtained ***the asymptotic model equations*** expressed only in ***macrodisplacements***. ***Coefficients in the asymptotic equations are constant, but they are independent of the microstructure cell size.*** Thus, ***contrary to the general tolerance models, the consistent asymptotic ones is not able to describe the length-scale effect on the overall shell dynamics.***

It has to be emphasized that the asymptotic models for the periodic shells under consideration obtained in this dissertation coincide with the corresponding asymptotic models proposed in [**Tomczyk B.:** *Length-scale effect in dynamics and stability of thin periodic cylindrical shells*. Scientific Bulletin of the Lodz University of Technology, No. 1166, series: Scientific Dissertations, Lodz University of Technology Press, Lodz **2013**]. It follows from the fact that in the consistent asymptotic averaging, the notion of weakly slowly-varying or slowly-varying functions is not introduced. In the asymptotic modelling, we deal only with the concepts of fluctuation shape functions and averaging operation.

***The two <u>new</u> general combined asymptotic-tolerance models*** for the analysis of selected dynamic problems in the biperiodic or uniperiodic shells under consideration have been formulated by applying ***the general asymptotic-tolerance modelling technique.***

This combined modelling includes ***the consistent asymptotic and the general tolerance non-asymptotic modelling techniques,*** which are combined together into a new procedure. ***The general combined model equations proposed here consist of the asymptotic (macroscopic)***

***model equations formulated by means of the consistent asymptotic procedure and having constant coefficients independent of a microstructure length and of the general superimposed tolerance (microscopic) model equations derived by applying the extended tolerance modelling technique and having constant coefficients depending also on a cell size. The asymptotic and tolerance models are combined together under assumption that in the framework of the asymptotic model the solutions to the problem under consideration are known.*** It has been shown *that under special condition imposed on the fluctuation shape functions,* ***the general combined models make it possible to separate the macroscopic description of some special problems from their microscopic description.*** Thus, ***an important advantage of the models is that they allows us to study micro-dynamics of the shells under consideration independently of the shells' macro-dynamics.***

*The macroscopic equations* formulated by means of *the consistent asymptotic modelling* and being independent of the periodicity cell size are identical for both the general and standard combined biperiodic or uniperiodic shell models. Both the general and standard asymptotic-tolerance models have ***constant coefficients***. Both the general and standard ***microscopic equations*** derived in the second step of the combined modelling ***depend on a cell size.*** Hence, both the models allows us to investigate *the length-scale effect in dynamical problems. General combined model equations* contain a bigger number of terms depending on the microstructure size than *the standard combined model equations*. In the framework of ***microscopic part of the general combined model for biperiodic shells***, unknown ***weakly slowly-varying fluctuation (microscopic) amplitudes*** are governed by ***partial differential equations,*** whereas within the framework of ***microscopic part of the standard combined model***, unknown ***slowly-varying fluctuation amplitudes*** are governed by ***ordinary differential equations*** involving only time derivatives of fluctuation amplitudes. Both the general and standard combined models for biperiodic shells make *it possible to describe selected problems of the shell micro-dynamics independently of the shell macro-dynamics.* ***A very important difference between the considered models is that the microscopic equations of the general combined model for biperiodic shells can be applied to the analysis of certain initial-boundary layer and space-boundary layer phenomena strictly related to the specific form of initial and boundary conditions imposed on the micro-fluctuation amplitudes, whereas microscopic equations of the standard combined model for biperiodic shells can only be applied to the analysis of certain initial-boundary layer phenomena. Microscopic equations of the general combined model for biperiodic shells describe the length-scale effect not only in non-stationary but also in stationary problems, whereas microscopic equations of the standard combined model make it possible to investigate the length-scale effect only in non-stationary problems.*** On the other hand, the macroscopic equations of both the general and standard combined models for the considered ***uniperiodically heterogeneous shells*** include terms with space and time derivatives of the fluctuation amplitudes. Hence, ***the equations of both uniperiodic shell models describe certain time-boundary and space-boundary layer phenomena.***

***The general and standard tolerance models and the asymptotic models for the biperiodic and uniperiodic shells under consideration have been applied to evaluation of the length-scale effect in some special problems dealing with free vibrations of the considered periodic shells.*** It has been shown that ***in the framework of the general and standard tolerance models, not only the fundamental cell-independent lower, but also the new additional higher-order cell-dependent free vibration frequencies can be derived and analysed. The higher free vibration frequencies cannot be determined applying asymptotic models commonly used for investigations of dynamics of the periodic shells.*** From both the analytical and computational results it follows that ***the differences between values of fundamental lower free vibration frequencies derived from the standard tolerance models for biperiodic or uniperiodic shells and values of free vibration obtained from the asymptotic ones are negligibly small.*** Moreover, from the computational results it follows that ***the differences between values of fundamental lower free vibration frequencies derived from the standard tolerance models for biperiodic or uniperiodic shells and values of free vibration frequencies obtained from the general tolerance ones are also negligibly small.*** Thus, ***the effect of the periodicity cell size on the fundamental lower free vibration frequencies of the shells under consideration can be neglected***. Hence, ***the asymptotic models being more simple than the tolerance non-asymptotic ones are sufficient from the point of view of calculations made for the dynamic problems under consideration.***

***The general microscopic equations for biperiodic shells and for uniperiodic shells have been applied to the analysis of the length-scale effect in some special problems dealing with the cell-dependent free micro-vibrations and with the long waves propagating in the unbounded shells along axial direction.*** These micro-dynamic equations derived in the second step of the general combined asymptotic-tolerance modelling are independent of solutions obtained in the framework of the macroscopic (i.e. asymptotic) models formulated in the first step of the combined modelling. Hence, ***they allow us to analyse selected problems of the shell micro-dynamics independently of the shell macro-dynamics. This is the greatest advantage of the proposed general combined models.***

***Applying the general microscopic equations for biperiodic shells and for uniperiodic shells, the cell-dependent free micro-vibration frequencies have been derived independently of the cell-independent free macro-vibration frequencies.*** The dependence of these frequencies on the microstructure length parameter was studied. The effects of the differences between the elastic properties and also between the inertial properties of the component materials in the cell on these frequencies were investigated in detail. ***Formulae for the free micro-vibration frequencies obtained in the framework of the microscopic parts of the general combined models contain a bigger number of terms depending on a cell size than corresponding expressions for the free micro-vibration frequencies derived from the microscopic parts of the***

***standard ones.*** Thus, from the analytical results it follows that the extended microscopic models make it possible to describe the length-scale effect more detail than the standard ones. ***However, from the numerical results it follows that in the micro-dynamic problem under consideration the differences between values of the corresponding free micro-vibration frequencies derived from the general and standard combined models are negligibly small. Hence, the simpler standard combined models are sufficient to determine and study the cell-dependent free micro-vibration frequencies of the periodic shells under consideration.***

***Applying the general microscopic equations for biperiodic shells,*** a special length-scale problem of axial harmonic micro-vibrations with vibration frequency $\tilde{\omega}$ has been discussed. ***Some new important results have been obtained.*** It has been shown that the shape of these micro-vibrations depends on relations between values of frequency $\tilde{\omega}$ and ***a certain new additional cell-dependent free vibration frequency*** $\tilde{\omega}_*$. ***The micro-vibrations can decay exponentially. They can decay linearly. For certain interrelations between*** $\tilde{\omega}$ ***and*** $\tilde{\omega}_*$ ***we deal with a non-decayed form of micro-vibrations (micro-vibrations oscillate) or with resonance micro-vibrations.*** Moreover, ***it has been shown that the micro-dynamic equations of the general combined model for biperiodic shells describe the space-boundary layer phenomena. It has to be emphasized that this micro-dynamic problem can be analysed neither in the framework of the asymptotic models which neglect the length-scale effect nor within the standard combined models for the biperiodic shells, in which unknown fluctuation amplitudes are governed by ordinary differential equations involving only time derivatives.***

***The general microscopic equations for biperiodic shells and for uniperiodic shells have been applied to the analysis of the long wave propagation problems related to micro-fluctuations in axial direction.*** We deal with long waves when the characteristic length dimension of the cell is much smaller than the wavelength. The effect of a microstructure size on the shape of the displacement waves and on the *wave propagation speed* has been shown. The influence of the differences between the elastic, inertial and geometrical properties of the constituents in the periodicity cell on the wave propagation speed has been investigated in detail. ***Some new important results have been obtained.*** It has been shown that ***the periodic microheterogeneity of the shells leads to exponential waves and to dispersion effects.*** Moreover, ***the new wave propagation speed depending on the microstructure size has been obtained.*** The highest values of the cell-dependent wave propagation speed have been obtained for periodic shells with a very strong inertial heterogeneity and with elastic homogeneous structure. The smallest values of this speed have been obtained for periodic shells with a very strong elastic heterogeneity and with inertial homogeneous structure. ***All the effects mentioned above cannot be analysed by applying the asymptotic models, which neglect the length-scale effect. Moreover, exponential waves and dispersion effects in the biperiodic shell under consideration cannot be studied in the framework of standard combined model for biperiodic shells.*** Micro-dynamic equation of ***the standard combined model*** describing micro-dynamic behaviour of ***the biperiodic shells*** in axial direction does not contain spatial derivatives of micro-fluctuation amplitude; it contains only time derivatives of this fluctuation amplitude. The above effects cannot be analysed by using this ordinary differential equation. On the other hand, exponential waves and dispersion effects in ***the uniperiodic shells*** can be investigated by applying both the general and standard combined models. In the problem to be examined, the extended model for uniperiodic shells overlaps with the standard model for these shells.

The governing equations of all models derived in this dissertation are uniquely determined by the periodic, continuous and highly oscillating ***fluctuations shape functions*** representing oscillations inside a cell. These functions are assumed to be known in every problem under consideration. They can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell. These functions can also be regarded as *the shape functions* resulting from the periodic discretization of the cell using for example the finite element method. The choice of these functions may also be based on the experience or intuition of the researcher.

Solutions to the initial-boundary value problems formulated in the framework of general tolerance models and microscopic parts of the general combined models have a physical sense only if ***the basic unknowns of these models are weakly slowly-varying functions with respect to the periodicity cell and pertinent tolerance parameters.*** This requirement can be verified only *a posteriori* and it imposes certain restrictions on the class of problems described by the general tolerance and general asymptotic-tolerance models proposed in this dissertation.

It should be noted that the general tolerance model and the general combined model (i.e. part of the combined model derived by means of tolerance modelling) for uniperiodic shells are not special cases of the corresponding models for biperiodic shells. General models of uniperiodic shells and those of biperiodic shells have to be led out independently because the modelling physical reliability conditions for uniperiodic shells are hold only in one periodicity direction, whereas for biperiodic shells these conditions are hold in two periodicity directions tangent to the shell midsurface. For uniperiodic shells we deal with functions which are ***weakly slowly-varying*** or ***tolerance-periodic*** or ***periodic*** in only ***circumferential direction***, whereas for biperiodic shells these functions are weakly slowly-varying or tolerance-periodic or periodic in ***circumferential and axial directions.***

The periodic shells being objects of considerations in this doctoral dissertation are widely applied in civil engineering, most often as roof girders and bridge girders. They are also widely used as housings of reactors and tanks. Periodic shells having small length dimensions are elements of air-planes, ships and machines.

***The results obtained in this dissertation have an essential influence on the state of knowledge dealing with dynamic behaviour of thin-walled periodically microheterogeneous cylindrical shells. The results also generate new directions of further investigations.*** The anticipated directions of further investigations can be related to: the general tolerance modelling of stationary and dynamic stability problems, the general tolerance modelling of geometrically non-linear dynamic and stability problems, the general tolerance modelling of dynamic thermoelasticity problems, the general tolerance modelling of mechanical and thermo-mechanical problems for cylindrical shells with a tolerance-periodic microstructure and functionally graded macrostructure and others.

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